Yang-Baxter equation for the asymmetric eight-vertex model

W. Galleas and M. J. Martins
Departamento de Física, Universidade Federal de São Carlos, Caixa Postal 676, 13565-905, São Carlos, Brazil
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In this paper we study, in the manner of Baxter [R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, New York, 1982)], the possible integrable manifolds of the asymmetric eight-vertex model. As expected, they occur when the Boltzmann weights are either symmetric or satisfy the free-fermion condition; but our analysis clarifies the reason why both manifolds need to share a universal invariant. We also show that the free-fermion condition implies three distinct classes of integrable models.

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Exactly solved vertex models play a fundamental role in classical statistical mechanics. The most important of these is the so-called eight-vertex model that contains as special cases most systems on a plane square lattice [1]. The general asymmetric eight-vertex model possesses six different Boltzmann weights $a_{\pm}$, $b_{\pm}$, c, and d whose transfer matrix can be written as

$$T = \Gamma r_2[\mathcal{L}_L \cdots \mathcal{L}_1], \tag{1}$$

where the trace is over the ordered product of local operators $\mathcal{L}_j$ that are given by the following 2×2 matrix:

$$\mathcal{L}_j = \begin{pmatrix} a_+\sigma_j^+\sigma_j^- + b_+\sigma_j^-\sigma_j^+ & d\sigma_j^+ + c\sigma_j^- \\ c\sigma_j^+ + d\sigma_j^- & b_-\sigma_j^+\sigma_j^- + a_-\sigma_j^-\sigma_j^+ \end{pmatrix} \tag{2}$$

and $\sigma_j^{\pm}$ are Pauli matrices acting on sites j of a one-dimensional lattice. The asymmetric eight-vertex model is known to be solvable in the manifolds.

$$F(a_{\pm},b_{\pm},c,d) = 0, \quad \frac{cd}{a_+b_- + a_-b_+} = I_1^F,$$

$$\frac{a_+^2 + b_-^2 - a_-^2 - b_+^2}{a_+b_- + a_-b_+} = I_2^F; \tag{3}$$

$$a_{\pm} = b_{\mp}, \quad c = d \quad \text{or} \quad a_{\pm} = b_{\pm}, \quad c = d; \tag{4}$$

$$a_+ = a_-, \quad b_+ = b_-, \quad \frac{cd}{a_+b_+} = I_1^B, \quad \frac{F(a_{\pm},b_{\pm},c,d)}{a_+b_+} = I_2^B; \tag{5}$$

where $F(a_{\pm},b_{\pm},c,d) = a_+a_- + b_+b_- - c^2 - d^2$ and $I_{1,2}^{F,B}$ are arbitrary constants.

The manifold (3) is the so-called free-fermion model whose free-energy was first calculated by Fan and Wu [2] and later rederived by Felderhof [3] who devised a method to diagonalize the corresponding transfer matrix. The integrability of the free-fermion manifold is usually assumed from the fact that its transfer matrix commutes with the *XY* Hamiltonian as shown by Krinsky [4] who used a procedure first developed by Sutherland [5]. Later on Barouch [6] and Kasteleyn [7] have revisited the problem of commuting asymmetric eight-vertex transfer matrices and generalized Heisenberg Hamiltonians that led Kasteleyn [7] to point out the existence of the manifolds (4). As stressed by this author, however, such manifolds are trivial because they can be seen as a set of independent one-dimensional models and therefore they should be disregarded. On the other hand, the solution of the symmetric manifold was found by Baxter through quite general approach, denominated commuting transfer matrix method that culminated in the famous "star-triangle" relations [1]. The fact that the symmetric eight-vertex transfer matrix commutes with a related *XYZ* Hamiltonian [5] has then been made more precise because the latter is essentially a logarithmic derivative of the former [1].

It would be quite desirable to extend the Baxter method to the asymmetric eight-vertex model and to rederive the manifolds (3) and (5) from a unified point of view. Since this approach does not assume *a priori* the existence of a specific local form for the corresponding Hamiltonian, it can lead us to integrable manifolds not covered by the analysis of Barouch [6] and Kasteleyn [7]. We recall that much of the work on this problem, see, e.g., Refs. [8–10] has been concentrated on analyzing the Yang-Baxter equations directly in terms of spectral parameters. Though this is a valid approach, it often hides the general integrable manifolds in terms of specific parametrizations that need to be found by *a posteriori* guess work. A more direct way would be first to determine the solvable manifolds by an algebraic study in the manner Baxter of the corresponding star-triangle equations and afterwards to parametrize them by using the theory of uniformization of biquadratic polynomials [1]. It appears that Kasteleyn [7] was the first to make an effort toward such analysis, but the best he could do was to guess the manifold (3) from known results by Felderhof besides clarifying the origin of the pseudo-one-dimensional manifold (4) as the linearization of the Yang-Baxter equation around a nonidentity 4×4 R matrix. Since the later possibility leads us to trivial manifolds we will disregard it, as did Kasteleyn, from our forthcoming analysis.

The probable reason that such generalization has not yet been carried out seems technical, since in the asymmetric model we have to deal with the double number of equations as compared to the symmetric eight-vertex model. At first sight this appears to be a cumbersome task, but here we show that it is possible to simplify this problem, without take recourse to computer manipulations, to a number of simple equations that will clarify the common origin of the above two integrable manifolds. Besides that, this approach allows us to show that manifold (3) is one of the three possible different integrable branches satisfying the free-fermion condition. The star-triangle relations are sufficient conditions [1] for commuting transfer matrices and for the asymmetric eight-vertex model they are given by

$$a_{\pm}a'_{\pm}d''+dc'a''_{\mp}=cd'a''_{\pm}+b_{\mp}b'_{\mp}d'', \tag{6}$$

$$db'_{\pm}c''+a_{\pm}d'b''_{\mp}=b_{\pm}d'a''_{\pm}+cb'_{\mp}d'', \tag{7}$$

$$db'_{\pm}b''_{\pm}+a_{\pm}d'c''=da'_{\mp}a''_{\pm}+a_{\mp}c'd'', \tag{8}$$

$$ca'_{\pm}c''+b_{\pm}c'b''_{\mp}=a_{\pm}c'a''_{\pm}+da'_{\mp}d'', \tag{9}$$

$$ca'_{\pm}b''_{\pm}+b_{\pm}c'c''=cb'_{\pm}a''_{\pm}+b_{\mp}d'd'', \tag{10}$$

$$b_{\mp}a'_{\pm}c''+cc'b''_{\mp}=dd'b''_{\pm}+a_{\pm}b'_{\mp}c''. \tag{11}$$

Note that each of these equations possesses two possibilities and we shall denote them by Eqs. (6)–(11). Altogether we have twelve linear homogeneous equations and only six weights, say $a''_{\pm}$, $b''_{\pm}$, c'' and d'', are at our disposal to be eliminated in terms of the remaining set of weights $\{a_{\pm},b_{\pm},c,d\}$ and $\{a'_{\pm},b'_{\pm},c',d'\}$. Therefore we have to choose the appropriate equations to start with and our solution goes as follows. We first eliminate the weights $a''_{\pm}$ with the help of the pair of Eqs. (9) and (10) and by substituting the result in Eqs. (6) we find the following relations:

$$\begin{aligned} b''_{\pm}(a_{\mp}ca'_{\pm}d'-db_{\mp}b'_{\pm}c') &= c''(cda'_{\mp}b'_{\pm}-a_{\mp}b_{\pm}c'd') \\ &\quad +d''[a_{\mp}b_{\mp}(d'^2-b'_{\pm}b'_{\mp}) \\ &\quad +a'_{\pm}b'_{\pm}(a_{\pm}a_{\mp}-d^2)]. \end{aligned} \tag{12}$$

Next we apply a similar procedure in the case of Eqs. (7) and (8) and the corresponding relations between the weights $b''_{\pm}$, c'', and d'' are

$$\begin{aligned} b''_{\pm}(db_{\mp}b'_{\pm}c'-a_{\mp}ca'_{\pm}d') &= c''(cda'_{\pm}b'_{\mp}-a_{\pm}b_{\mp}c'd') \\ &\quad +d''[a_{\mp}b_{\mp}(c'^2-a'_{\pm}a'_{\mp}) \\ &\quad +a'_{\pm}b'_{\pm}(b_{\pm}b_{\mp}-c^2)]. \end{aligned} \tag{13}$$

From Eqs. (12) and (13) it is not difficult to eliminate the weights $b''_{\pm}$, leading us to constraints between c'' and d'',

$$\begin{aligned} &c''[cd(a'_{\pm}b'_{\mp}+a'_{\mp}b'_{\pm})-c'd'(a_{\pm}b_{\mp}+a_{\mp}b_{\pm})] \\ &\quad =d''[a_{\mp}b_{\mp}F(a'_{\pm},b'_{\pm},c',d')-a'_{\pm}b'_{\pm}F(a_{\pm},b_{\pm},c,d)]. \end{aligned} \tag{14}$$

At this point it is tempting to use such equations and the previous results for $a''_{\pm}$ and $b''_{\pm}$ to eliminate the ratios of the five weights and to substitute them in the remaining equations, namely, Eqs. (11), and either Eqs. (12) or Eqs. (13). This is, however, not so illuminating because it leads us to carry out simplifications in complicated expressions. We find that it is more profitable to repeat the procedure described above, but now we first eliminate the weights $b''_{\pm}$ and in the end we use Eqs. (11) instead of Eqs. (6). This leads us to a different constraint between c'' and d'' given by

$$\begin{aligned} &d''[cd(a'_{\pm}b'_{\mp}+a'_{\mp}b'_{\pm})-c'd'(a_{\pm}b_{\mp}+a_{\mp}b_{\pm})] \\ &\quad =c''[a_{\mp}b_{\mp}F(a'_{\pm},b'_{\pm},c',d')-a'_{\mp}b'_{\mp}F(a_{\pm},b_{\pm},c,d)]. \end{aligned} \tag{15}$$

Now we reached a point that enables us to make conclusions on the way the set of weights $\{a_{\pm},b_{\pm},c,d\}$ and $\{a'_{\pm},b'_{\pm},c',d'\}$ should be related to each other. In fact, from Eqs. (14) and (15) we find that the necessary conditions for the weights c'' and d'' not to be all zero are

$$\frac{cd}{a_+b_-+a_-b_+}=\frac{c'd'}{a'_+b'_-+a'_-b'_+} \tag{16}$$

and either

$$F(a_{\pm},b_{\pm},c,d)=F(a'_{\pm},b'_{\pm},c',d') \tag{17}$$

or

$$\frac{a_+b_+}{a_-b_-}=\frac{a'_+b'_+}{a'_-b'_-}=1, \quad \frac{F(a_{\pm},b_{\pm},c,d)}{a_-b_-}=\frac{F(a'_{\pm},b'_{\pm},c',d')}{a'_-b'_-}. \tag{18}$$

We are already in a position to conclude that the asymmetric eight-vertex model has indeed only two possible integrable manifolds, one is singled out by the free-fermion condition (17) while the other, Eq. (18), turns out to be a mixed type of conditions that relate the set of weights both alone and between each other. One important point of our analysis is that it makes clear that both manifolds need to share a common invariant given by Eq. (16).

To conclude our analysis, it remains to check the consistency between Eqs. (6) and Eqs. (11), which can in principle be a source of further constraints. From such equations one can easily calculate the ratios a''_+/a''_- and b''_+/b''_-, namely,

$$\frac{a''_+}{a''_-}=\frac{cd'(a_+a'_+-b_-b'_-)-dc'(b_+b'_+-a_-a'_-)}{dc'(a_+a'_+-b_-b'_-)-cd'(b_+b'_+-a_-a')}, \tag{19}$$

$$\frac{b''_+}{b''_-}=\frac{dd'(b_-a'_+-a_-b'_-)-cc'(a_-b'_+-b_+a'_-)}{cc'(b_-a'_+-a_+b'_-)-dd'(a_-b'_+-b_+a'_-)}, \tag{20}$$

which in principle can be compared with our previous results for the same ratios.

Before proceeding with that, however, there exists one property that we have not yet explored. Instead of starting our analysis by eliminating the weights $a''_{\pm}$, $b''_{\pm}$, c'' and d'' we could choose to begin with the other two sets of weights as well. Because the star-triangle equations are not symmetric by exchanging a given two sets of weights, we expect that each possibility will lead us to a different kind of constraints. This means that we can use the asymmetry of the weights in our favor, which may help us in further simplifications. For example, the relations (6)–(11) are invariant under the exchange of weights $\{a''_{\pm},b''_{\pm},c'',d''\}$ and $\{a_{\pm},b_{\pm},c,d\}$ only after the transformation $b_{\pm}\rightarrow b_{\mp}$ is performed for all set of weights. This means that if we had started our procedure by eliminating the weights $a_{\pm}$, $b_{\pm}$, c, and d the same analysis we have carried out so far will lead us to the following constraints:

$$\frac{c'd'}{a'_+b'_++a'_-b'_-}=\frac{c''d''}{a''_+b''_++a''_-b''_-}, \tag{21}$$

besides that either

$$F(a'_{\pm},b'_{\pm},c',d')=F(a''_{\pm},b''_{\pm},c'',d'')=0 \tag{22}$$

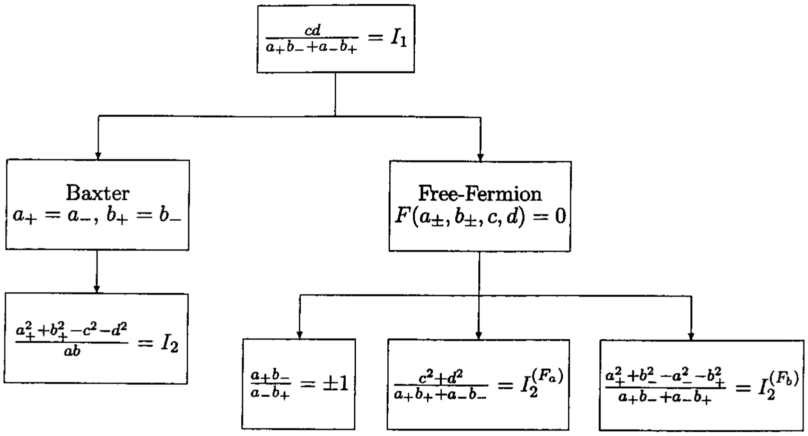

FIG. 1. Summary of the integrable manifolds of the asymmetric eight-vertex model. The symbols I_1, $I_2^{(F_a)}$, and $I_2^{(F_b)}$ denote invariants for two distinct sets of weights.

or

$$\frac{a'_-b'_+}{a'_+b'_-} = \frac{a''_-b''_+}{a''_+b''_-} = 1,$$

$$\frac{F(a'_\pm, b'_\pm, c', d')}{a'_-b'_+} = \frac{F(a''_\pm, b''_\pm, c'', d'')}{a''_-b''_+}. \tag{23}$$

By the same token if we had started by eliminating $a'_\pm$, $b'_\pm$, c', and d' we will find

$$\frac{cd}{a_+b_++a_-b_-} = \frac{c''d''}{a''_+b''_-+a''_-b''_+} \tag{24}$$

and that either

$$F(a_\pm, b_\pm, c, d) = F(a''_\pm, b''_\pm, c'', d'') = 0 \tag{25}$$

or

$$\frac{a_-b_+}{a_+b_-} = \frac{a''_+b''_+}{a''_-b''_-} = 1, \quad \frac{F(a_\pm, b_\pm, c, d)}{a_-b_+} = \frac{F(a''_\pm, b''_\pm, c'', d'')}{a''_-b''_-}. \tag{26}$$

Let us now analyze the consequences of this observation for each possible integrable manifold and here we begin with the second manifold. It is not difficult to see that the consistency of Eqs. (18), (23), and (26), to what concern relations within the same set of weights, impose severe restrictions on the second type of the manifold, namely,

$$a_+ = a_- \text{ and } b_+ = b_- \text{ or } a_+ = -a_- \text{ and } b_+ = -b_- \tag{27}$$

and similar conditions for the other sets $\{a'_\pm, b'_\pm\}$ and $\{a''_\pm, b''_\pm\}$.

It turns out, however, that the only possibility compatible with the "universal" constraints (16), (21), and (24) is the totally symmetric case $a_+ = a_-$ and $b_+ = b_-$ leading us therefore to the Baxter model (5). Note that in this situation the compatibility between Eqs. (6) and Eqs. (11) is trivial because both Eqs. (19) and (20) are automatically satisfied.

We now turn our attention to the free-fermion manifold. In this case we have much less restrictive constraints since we are only left with relations between different weights, namely, Eqs. (16), (21), and (24). Altogether these equations provide us the following relation:

$$\frac{a''_+b''_+ + a''_-b''_-}{a''_+b''_- + a''_-b''_+} = \frac{a_+b_- + a_-b_+}{a_+b_+ + a_-b_-} \, \frac{a'_+b'_+ + a'_-b'_-}{a'_+b'_- + a'_-b'_+} \tag{28}$$

whose compatibility with Eqs. (6) and (11) can be implemented by evaluating the left-hand side of Eq. (28) with the help of Eqs. (19) and (20). After few manipulations, in which the free-fermion condition is explicitly used, we end up with a "separable" equation $P=0$ for the weights $\{a_\pm, b_\pm, c, d\}$ and $\{a'_\pm, b'_\pm, c', d'\}$ and the polynomial P is given by

$$\begin{aligned} P = {} & [(c^2+d^2)(a'_+b'_+ + a'_-b'_-) - (a_+b_+ + a_-b_-) \\ & \times (c'^2+d'^2)][a_-b_+a'_-b'_+ - a_+b_-a'_+b'_-] \\ & \times [(a_+^2+b_-^2-a_-^2-b_+^2)(a'_+b'_- + a'_-b'_+) \\ & -(a_+b_- + a_-b_+)(a'^2_+ + b'^2_- - a'^2_- - b'^2_+)]. \end{aligned} \tag{29}$$

From this equation we conclude that we have three possible free-fermion integrable manifolds given by either

$$\frac{a_+^2+b_-^2-a_-^2-b_+^2}{a_+b_-+a_-b_+} = \frac{a'^2_+ + b'^2_- - a'^2_- - b'^2_+}{a'_+b'_- + a'_-b'_+} \tag{30}$$

or

$$\frac{a_+b_-}{a_-b_+} = \frac{a'_+b'_-}{a'_-b'_+} = \pm 1, \tag{31}$$

or still

$$\frac{c^2+d^2}{a_+b_++a_-b_-} = \frac{c'^2+d'^2}{a'_+b'_+ + a'_-b'_-}, \tag{32}$$

besides, of course, the free-fermion conditions for both $\{a_\pm, b_\pm, c, d\}$ and $\{a'_\pm, b'_\pm, c', d'\}$ together with the "universal" relation (16). Note that the free-fermion case (31) cannot be related to the manifold (4) beginning by the fact that in the former model the weight c can be different than the weight d.

In Fig. 1 we have summarized all the results obtained so far. Let us now compare our results with previous work in the literature. Contrary to what happened to the symmetric manifold (5) we recall that Eqs. (30) and (31) do not imply that the ratios $(a_+^2+b_-^2-a_-^2-b_+^2)/(a_+b_-+a_-b_+)$ and $(c^2+d^2)/(a_+b_++a_-b_-)$ are necessarily constants, but only that they are invariants for two distinct sets of weights.[1] This is the reason why general solutions of the Yang-Baxter equation satisfying the free-fermion condition are expected to be nonadditive [10]. In fact, in the Appendix we show that the additional assumption of additivity provides us extra restriction to the weights. In this sense, the manifold (30) turns out to be a generalization of the original result (3) by Krinsky [4]. Next the manifold (31) has been only partially obtained in the literature, more precisely in the special case $a_+ = a_-$ and $b_+ = -b_-$ [8,10].[2] Finally, to the best of our knowledge, the last branch (32) is new in the literature. The probable reason why such general manifolds have been missed in previous work, see, for example, Refs. [8,10], is related to the analysis of the Yang-Baxter equation in terms of spectral parameters. There it was required that a certain value of the spectral parameter (initial condition) weights should be regular, i.e., that the corresponding L_j operator be proportional to the four-dimensional permutator. Note that the L_j operator of manifold (32) cannot be made regular and therefore does not have a local associated Hamiltonian. This is also the reason why Barouch [6] and Kasteleyn [7] missed such manifold since they used the assumption of local forms of Hamiltonians. We recall that though the property of regularity guarantees that the logarithmic derivative of the transfer matrix is *local*, this is by no means a necessary condition for integrability.

In summary, we have analyzed according to Baxter the integrable branches of the asymmetric eight-vertex model. Besides recovering Baxter's model we have shown that the free-fermion condition produces three different sets of integrable manifolds. A natural question to be asked is whether or not the new manifolds (31) and (32) can be solved by the method devised by Felderhof originally proposed to diagonalize the transfer matrix of Krinsky's manifold (3). This is of interest since these systems can be the corner stone of highly nontrivial models as have been recently discussed in Refs. [11,12]. In fact, we have evidences that the manifold (31) is related to a staggered *XY* model. Because both the Baxter symmetric model and the free-fermion manifolds (30)–(32) share a common algebraic structure, the Yang-Baxter algebra, it is plausible to think that Baxter's generalized Bethe ansatz can be adapted to include the solution of the free-fermion models too. This problem has eluded us so far though some progress has been made in the case of the simplest free-fermion branch (31).

[1]The same statement is of course valid for the "universal" ratio $cd/(a_+b_- + a_-b_+)$.

[2]Of course the other possibility $a_+ = a_-$ and $b_+ = b_-$ is contained in the Baxter solution.

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APPENDIX

The purpose here is to demonstrate that the hypothesis of additivity of the weights leads us to much more restrictive conditions for the free-fermion manifold as compared with the results (30)–(32) of the main text. In order to see that, lets us consider as usual that the weights $a_\pm$, $b_\pm$, c, d are parametrized by variables x_1 and x_2 and similarly that $a'_\pm$, $b'_\pm$, c', d' and $a''_\pm$, $b''_\pm$, c'', d'' are parametrized by x_1, x_3 and x_2, x_3, respectively. The consistency between the universal relations (16), (21), and (24) implies a remarkable separability condition for the ratio

$$\frac{a_+(x_1,x_2)b_-(x_1,x_2) + a_-(x_1,x_2)b_+(x_1,x_2)}{a_+(x_1,x_2)b_+(x_1,x_2) + a_-(x_1,x_2)b_-(x_1,x_2)} = \frac{G(x_1)}{G(x_2)}, \tag{A1}$$

where $G(x)$ is an arbitrary function.

The additional assumption that the weights are additive means that this function is necessarily a constant, which ultimately leads us to the relation

$$(a_+ - a_-)(b_+ - b_-) = 0. \tag{A2}$$

As a consequence of that, the possible manifolds satisfying the free-fermion condition are either $a_+ = a_-$ or $b_+ = b_-$. Now by imposing the consistency between Eq. (6) and Eq. (11) it turns out that these two possibilities becomes either

$$a_+ = a_- \quad \text{and} \quad b_+ = -b_- \tag{A3}$$

or

$$b_+ = b_- \quad \text{and} \quad \frac{a_+ - a_-}{b_+} = \Delta, \tag{A4}$$

where Δ is a constant.

Clearly, these are special cases of the manifolds (31) and (30), respectively.

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